

# $\tau$ Decay and the QCD infrared fixed point

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## **Abstract:**

We apply the optimization procedure based on the Principle of Minimal Sensitivity to the third-order calculation of  $R_\tau$ . Since the effective couplant remains finite, freezing to a value  $\alpha_s/\pi = 0.26$  at low energies, we can actually evaluate the defining integral of  $R_\tau$  and compare the optimized perturbation theory result to that of the optimized result obtained after the integral has been evaluated using contour techniques. The good agreement shows that the optimization procedure is consistent and suggests that the infrared fixed point is meaningful.

# 1 Introduction

The inclusive semihadronic decay rate of the  $\tau$  lepton, expressed as the ratio

$$R_\tau = \frac{\Gamma(\tau^- \rightarrow \nu_\tau + \text{hadrons})}{\Gamma(\tau^- \rightarrow \nu_\tau e^- \bar{\nu}_e)}, \quad (1)$$

is a fundamental test of QCD. The parton model gives a rough estimate by approximating hadronic decay as the rate into quark-antiquark pairs:  $R_\tau \approx N_c = 3$ . Corrections to this approximation include both perturbative and non-perturbative QCD, as well as electroweak corrections [1, 2]. Here we will concentrate on the perturbative QCD corrections and compare two different ways of applying the optimized perturbation theory to  $R_\tau$ .

Refs. [1, 2] discuss in detail the theoretical calculation of  $R_\tau$ , which has the following form:

$$R_\tau = \int_0^{M_\tau^2} ds D(s) = \int_0^{M_\tau^2} ds \frac{2}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left(1 + \frac{2s}{M_\tau^2}\right) \tilde{R}(s), \quad (2)$$

with

$$\tilde{R}(s) = 3(|V_{ud}|^2 + |V_{us}|^2)[1 + \mathcal{R}_\tau(s)]. \quad (3)$$

Using the couplant  $a \equiv \frac{\alpha_s}{\pi}$  the perturbative corrections are expressed as a series, truncated to third-order:

$$\mathcal{R}_\tau = a(1 + r_1 a + r_2 a^2). \quad (4)$$

From the recently re-done third-order calculation of Gorishny, *et al.* [3, 4] :

$$r_1(\overline{\text{MS}}, \mu = M_\tau) = 1.986 - 0.115n_f, \quad (5)$$

$$r_2(\overline{\text{MS}}, \mu = M_\tau) = -6.637 - 1.200n_f - 0.005n_f^2, \quad (6)$$

in the  $\overline{\text{MS}}$  scheme with the renormalization scale  $\mu$  taken to be  $M_\tau$ . For three flavors,  $r_1 = 1.64$  and  $r_2 = -10.28$ . [Note this differs greatly from the earlier result  $r_2 = 93.98$  [1] based on the erroneous calculation of [5].]

Historically the integral (2) has not been evaluated directly because it was expected that the QCD couplant would become large at small  $s$ , making a perturbative prediction impossible. Instead,  $R_\tau$  was re-expressed as a contour integral in the complex  $s$  plane with the contour running clockwise around the circle of  $|s| = M_\tau^2$ , thus avoiding the small- $s$  region. To evaluate the integral, the coupling constant is expanded in powers of  $\alpha_s(M_\tau)$  [1, 2]. The final result (for three flavors) is then, in the  $\overline{\text{MS}}(\mu = M_\tau)$  scheme, [3]:

$$R_\tau = 3(|V_{ud}|^2 + |V_{us}|^2)[1 + a + 5.20a^2 + 26.37a^3]. \quad (7)$$

## 2 Optimized Perturbation Theory

Based on the principle of minimal sensitivity, optimized perturbation theory (OPT) [6] finds the renormalization scheme (RS) in which the result is least sensitive to changes in the RS parameters. For a detailed discussion of the OPT method see [6]. The application of Ref. [7] to the  $R_{e^+e^-}$  ratio in third order is easily adapted to the present case. In effect, the difference between QCD corrections to the ratio of the  $e^+e^-$  hadronic cross section and the  $\tau$  hadronic decay involves taking  $(\sum Q_f)^2 = 0$  (in  $r_2(\overline{\text{MS}})$ ) and replacing  $3\sum Q_f^2$  by  $3\sum V_{ff}^2 \approx 3$ .

As previously explained in [8] the question of an infrared fixed point can be addressed by solving the optimization equations and requires a third-order calculation to determine the RS invariant  $\rho_2$ , where

$$\rho_2 \equiv r_2 + c_2 - (r_1 + \frac{1}{2}c)^2, \quad (8)$$

and  $c$  and  $c_2$  are  $\beta$ -function coefficients;

$$\mu \frac{\partial a}{\partial \mu} \equiv \beta(a) = -ba^2(1 + ca + c_2a^2 + \dots). \quad (9)$$

If  $\rho_2$  is negative a positive fixed point,  $\bar{a}^*$ , exists, and the more negative  $\rho_2$  is, the smaller that  $\bar{a}^*$  will be. Just as in the  $R_{e^+e^-}$  case, here  $\rho_2$  is negative and an infrared fixed point exists. Our analysis shows that the numerical solutions to the optimization equations tend towards that fixed-point solution. Because of this “freezing” of the coupling constant to a small finite value, the perturbative expansion of  $\mathcal{R}_\tau(s)$  does not diverge and the integral (2) can be evaluated numerically. Thus there are two ways of applying optimized perturbation theory to the problem of evaluating  $R_\tau$ : (i) optimization of the integrated expression for  $R_\tau$ , *i.e.* Eq. (7), and (ii) optimization of the perturbative expansion inside the integral down to low energies and then integrating the result. The second method gives a perturbative prediction for the differential decay rate.

Method (i) involves applying optimized perturbation theory to the  $\overline{\text{MS}}$  result for  $R_\tau$ , (7), and was previously done by Chyla *et al.*, [9]. Here we take  $V_{ud} = 0.9747$  and  $V_{us} = 0.218$  from the particle data book [10]. For three flavors and  $s = M_\tau^2$  [using  $\Lambda_{\overline{\text{MS}}}^{(3)} = 280$  MeV [11], and  $M_\tau = 1.777$  GeV [12]] we get the following optimum values:  $\bar{a}^{(3)}(M_\tau) = 0.164$ ,  $\bar{r}_1 = -0.49$ ,  $\bar{r}_2 = 2.98$  and  $\bar{R}_\tau = 3.48$ .

Now we can compare this result to method (ii) where we have to solve the optimization equations for  $\mathcal{R}_\tau(s)$  in the range  $M_\tau^2 > s > 0$  and evaluate the integrand of Eq. (2) explicitly. Note that the perturbative expression to be optimized in this case is (4) which has different coefficients than (7). At  $s = M_\tau^2$  the relevant number of flavors is three (and again we will take  $\Lambda_{\overline{\text{MS}}}^{(3)} = 280$  MeV for comparison to method (i)) but at lower energies there are only two active flavors, so we must ‘match’ to a 2-flavor theory with an appropriate  $\Lambda$ . We follow the method outlined in [7] by adjusting  $\Lambda_{\overline{\text{MS}}}^{(2)}$  so that  $\mathcal{R}_\tau$  is continuous at the matching point, which we take to be  $\sqrt{s} = 2m_s$ , (with a current quark mass of  $199 \text{ MeV} \pm 33$  [10]). This

leads to a value of  $\Lambda_{\overline{MS}}^{(2)} = 250$  MeV. Another relevant point is that the term  $|V_{us}|^2$  must be set to zero below the  $s,u$  kinematic threshold at  $\sqrt{s} = m_s + m_u$ , since for  $\sqrt{s}$  below this value the virtual  $W$  can decay only to an  $u,d$  pair.

As with  $R_{e^+e^-}$ , solving the optimization equations is straightforward except for the slow convergence of the iteration method at low energies. Numerical solutions were obtained down to  $\sqrt{s} = 0.10$  GeV, where the result joins smoothly to the analytic result for the  $\sqrt{s} \rightarrow 0$  limit [8, 7]. In Fig. 1 we show the optimized couplant  $\bar{a}$  as a function of  $\sqrt{s}$ . Also shown is the optimized third-order correction  $\bar{\mathcal{R}}_\tau(s)$ . At zero energy we have the infrared-fixed point solution for two flavors. In Table 1 we list the infrared values for 2 and 3 flavors. [In the  $R_{e^+e^-}$  case, the results  $\rho_2$  and  $a^*$  are identical for 3 flavors (since  $(\sum Q_f) = 0$  for  $u,d,s$  quarks), but slightly different for 2 flavors ( $\rho_2 = -10.91, a^* = 0.263$ ).]

From the optimized result for  $R_\tau(s)$  we can obtain  $D(s)$  from Eqs. (2),(3). This is plotted in Fig. 2. Integrating this from 0 to  $M_\tau^2$  gives  $R_\tau = 3.44$ , which agrees well with the result 3.48 obtained with method (i).

$D(s)$  represents a perturbative prediction for the differential decay rate of  $d\Gamma(\tau^- \rightarrow \nu_\tau + \text{hadrons}(s))/ds$ , normalized by  $\Gamma(\tau \rightarrow \nu_\tau e^- \bar{\nu}_e)$ , for  $\tau$  decays to hadrons with an invariant mass of  $\sqrt{s}$  [13]. We do not expect this prediction to be right, of course. The corresponding experimental quantity would presumably show structure due to meson thresholds and resonances, particularly  $\pi$  and  $a_1$ . However our hypothesis is that the perturbative prediction is meaningful in that if both the data and the prediction are ‘smoothed’ in some suitable fashion then they will agree. This hypothesis is supported by the results of Ref.[7] which finds excellent agreement when Poggio-Quinn-Weinberg (PQW) smearing is applied to both the experimental data and the theoretical prediction for  $R_{e^+e^-}$ . We suggest that

PQW smearing applied to  $\tilde{R}(s)$ , related to  $D(s)$  by (2), will also give good agreement in the  $\tau$  case. Unfortunately, it does not seem possible to test this prediction at present, since existing  $\tau$ -decay data give inadequate information about neutral hadrons. It is a bold prediction because it implies that the vector and axial-vector contributions, although dominated by different hadrons, must become the same after smearing. While the smearing is quite drastic (since it must smooth out the peak structure in the data) the theory/experiment comparison it allows is quantitative and highly non-trivial.

### 3 Theoretical Uncertainties

To answer the question of how good the agreement is between the two methods we need to ask where the uncertainty lies in the calculation. Other than the choice of  $\Lambda_{\overline{MS}}^{(3)}$  (which was the same for both methods) there is only one uncertainty in method (i): the truncation of the perturbative series. Following the argument of [7] we estimate the error as  $|\bar{r}_2 \bar{a}^3|$ , and thus for method (i):

$$R_\tau^{(i)} = 3.48 \pm 0.04. \quad (10)$$

In method (ii) there are two sources of error: truncation of the perturbation series and uncertainty in the input parameters (the strange quark mass and  $\Lambda^{(2)}$ ). The series-truncation error was again estimated as  $|\bar{r}_2 \bar{a}^3|$  and then integrated as in Eq. (3). This had an uncertainty of about  $\pm 0.10$ .

As explained by Marciano [14] the  $\Lambda_{\overline{MS}}$  of our effective theories (massless quarks with different numbers of flavors) must be matched at the thresholds to correspond to a single, underlying “full QCD” theory. The exact procedure for doing this, though, is unknown; should one make  $\mathcal{R}$  continuous or  $\alpha_s$  continuous across the threshold? The differences

between the two procedures are small, only about 5 MeV, but we make the following estimate to gauge the size of error: if  $\Lambda_{\overline{MS}}^{(2)}$  increases to 260 MeV then  $R_\tau$  increases by only 0.002, which shows that the perturbative corrections are insensitive to this uncertainty in  $\Lambda_{\overline{MS}}^{(2)}$ . This agrees with the  $e^+e^-$  analysis, [7], which found that below the  $s$ -quark threshold  $R_{e^+e^-}$  was most sensitive to the fixed-point solution.

The estimated error on the current strange quark's mass is about 20%. If the mass of the strange quark is decreased to 166 MeV, the 3 flavor region now runs down to 332 MeV. The effects of this on  $\mathcal{R}_\tau$  are barely discernable and do not significantly modify the integral. The moving of the threshold also affects the value of  $\Lambda_{\overline{MS}}^{(2)}$ . For both limits of the strange quark mass the  $\Lambda_{\overline{MS}}^{(2)}$  increases to about 260 MeV. As we found above this also has little effect on the error. The net effect of changing the strange quark mass is estimated to be  $\pm 0.006$ . Therefore with method (ii) we estimate:

$$R_\tau^{(ii)} = 3.44 \pm 0.11. \quad (11)$$

As one can see the two methods agree well within the error estimates of method (ii).

## 4 Comparison to Experiment

$R_\tau$  can be found by measuring the leptonic branching fractions,  $B_e, B_\mu$  [2]

$$R_\tau^{exp,B} = 3.66 \pm 0.05. \quad (12)$$

Independently  $R_\tau$  can also be found by measuring the total decay rate:

$$R_\tau^{exp,\Gamma} \equiv \frac{\Gamma_\tau - \Gamma_{\tau \rightarrow e} - \Gamma_{\tau \rightarrow \mu}}{\Gamma_{\tau \rightarrow e}}$$

where  $\Gamma_{\tau \rightarrow l} = \Gamma(\tau^- \rightarrow \nu_\tau l^- \bar{\nu}_l)$  can be calculated theoretically with great accuracy because it is a purely electroweak process. Using a mean life time of  $1/\Gamma = \tau_0 = (0.3025 \pm 0.0059) \times$

$10^{-12}$ s Ref. [2] quotes  $R_\tau^{exp,\Gamma} = 3.32 \pm 0.12$ . Taking the average of these two values gives:

$$R_\tau^{exp} = 3.61 \pm 0.05. \quad (13)$$

To compare to experiment we need to estimate the size of error due to the choice of  $\Lambda_{\overline{MS}}^{(3)}$ . Increasing the value by 50 MeV to 330 MeV leads to a 2% increase in method (i) and 1% increase in method (ii). In Table 2 we list the actual values for the case of increasing and decreasing  $\Lambda_{\overline{MS}}^{(3)}$  by 50 MeV. Note that method (i) is more sensitive to these changes than method (ii).

It is generally accepted that non-perturbative corrections are small, decreasing  $R$  by only 0.5%  $R_\tau^{pert}$  [15]. Electroweak corrections are estimated to increase  $R$  by 2%. Therefore we get the following estimates for the two methods:

$$R_\tau^{(i)} = 3.48 - 0.02 + 0.07 = 3.53 \pm 0.12, \quad (14)$$

$$R_\tau^{(ii)} = 3.44 - 0.02 + 0.07 = 3.48 \pm 0.15. \quad (15)$$

Both methods give very good results, which are near the experimental average and well within the errors.

## 5 Conclusion

Using optimized perturbation theory we have shown that the infrared fixed point and the ‘freezing’ of the strong coupling constant at low energies can be used to evaluate the  $R_\tau$  integral directly. Comparison to the usual OPT method of optimizing the  $\overline{MS}$  result is excellent showing the consistency of OPT. Furthermore both methods give good agreement to the experimental value.



## Acknowledgements

I would like to thank Paul Stevenson and Ian Duck for helpful comments. This work was supported in part by the U.S. Department of Energy under Contract No. DE-FG05-92ER40717.

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$n_f$	$\rho_2$	$a^*$	$\mathcal{R}_\tau(0)$
2	-10.83	0.264	0.332
3	-12.21	0.244	0.303

**Table 1:** Infrared fixed point values for 2 and 3 flavors.

$\Lambda_{\overline{MS}}^{(3)}$ (MeV)	$\mathcal{R}_\tau^{(i)}$	$\mathcal{R}_\tau^{(ii)}$
330	$3.56 \pm 0.06$	$3.48 \pm 0.13$
280	$3.48 \pm 0.04$	$3.44 \pm 0.10$
230	$3.41 \pm 0.02$	$3.40 \pm 0.08$

**Table 2:** Comparison of methods (i) and (ii) for different  $\Lambda_{\overline{MS}}^{(3)}$ . Errors are estimated by  $|r_2 a^3|$ , integrated for method (ii).

**Fig. 1.** The optimized third-order results for  $\bar{a} = \alpha_s/\pi$  and  $\bar{\mathcal{R}}_\tau^{(3)}(s)$ . The vertical line indicates where the  $N_f = 2$  and  $N_f = 3$  effective theories are matched. Error bars for  $\bar{\mathcal{R}}_\tau^{(3)}(s)$  are  $\pm|\bar{r}_2\bar{a}^3|$ .

**Fig. 2.** Integrand of  $R_\tau$ ; *i.e.* the differential decay rate  $d\Gamma(\tau^- \rightarrow \nu_\tau + \text{hadrons}(s))/ds$  for decays into hadrons with invariant mass-squared  $s$ , divided by  $\Gamma(\tau^- \rightarrow \nu_\tau e^- \bar{\nu}_e)$ . The ‘jump’ at  $\sqrt{s} \approx 200$  MeV occurs because the term  $|V_{us}|^2 = 0$  below the  $s, u$  kinematic threshold.

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